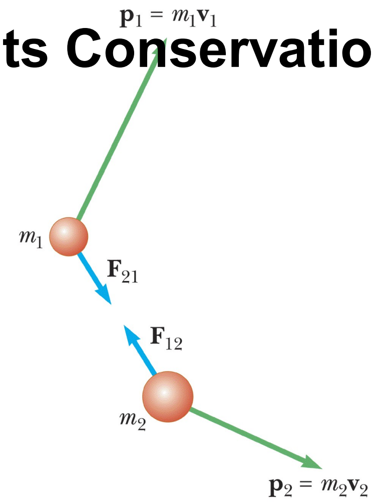
Lecture 09 Systems of Particles and Conservation of Linear Momentum

9.1 Linear Momentum and Its Conservation

9.2 Isolated System

Figure 8.1



linear momentum: $\vec{P} = m\vec{v}$

$$\sum \vec{F} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

isolated system

$$\vec{F}_{ext} = 0$$

$$\vec{F}_{21} = \frac{d\vec{P}_1}{dt}, \quad \vec{F}_{12} = \frac{d\vec{P}_2}{dt}$$

$$\vec{F}_{12} + \vec{F}_{21} = 0, \quad \frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = 0 = \frac{d}{dt}(\vec{P}_1 + \vec{P}_2)$$

$$\vec{P}_{tot} = const \rightarrow p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

the law of conservation of linear momentum

Example: Can we really ignore the kinetic energy of the Earth?

$$\frac{K_E}{K_b} = \frac{m_E v_E^2}{m_b v_b^2}, \quad m_E v_E = m_b v_b, \quad \frac{v_E}{v_b} = \frac{m_b}{m_E}, \quad \frac{K_E}{K_b} = \frac{m_E}{m_b} \cdot \frac{m_b^2}{m_E^2} = \frac{m_b}{m_E}$$

9.3 Nonisolated System (Impulse)

$$\vec{I} = \sum \vec{F} \cdot \Delta t = \Delta \vec{P}$$

Example: How good are the bumpers?

In a crash test, an automobile of mass 1500 kg collides with a wall. The initial and final velocities of the automobile are v_i = -15 m/s and v_f = 2.6 m/s. If the collision lasts for 0.15 s, find the impulse due to the collision and the average force excerted on the automobile.

$I = 1500 \cdot 2.6 - 1500 \cdot (-15) = 2.64 \cdot 10^4$, $F = \frac{I}{\Delta t} = \frac{2.64 \cdot 10^4}{0.15} = 1.76 \cdot 10^5 N$

9.4 Collisions in One Dimension

What is a collision?

A **collision** is an isolated event in which two or more bodies (the colliding bodies) exert relatively strong forces on each other for a relatively short time.

Don't need real touch.

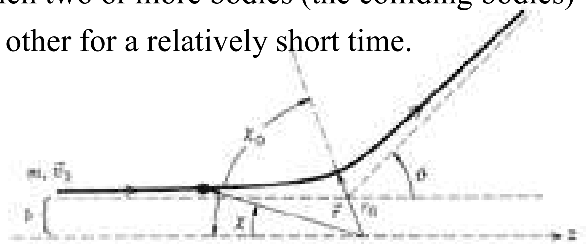
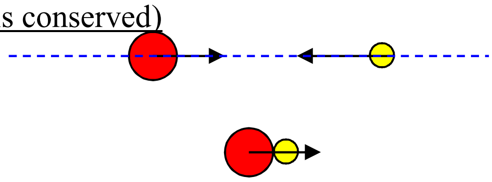


Figure 1.2 The scattering of a particle of mass m, with initial (asymptotic) velocity $\bar{v}_0$, from a center of force at the origin.

Rules:
The linear momentum must be conserved whether it is elastic or inelastic collision.
Elastic collision: the total kinetic energy must be conserved.
Inelastic collision: the total kinetic energy of the system is not the same before and after the collision.

Perfectly Inelastic Collisions: (linear momentum is conserved)

$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f$, $v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$



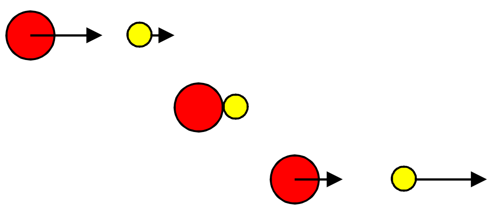
Elastic Collisions: (both linear momentum and energy are conserved)

$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$, $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2$

$m_1 (v_1^2 - v'^2_1) = m_2 (v'^2_2 - v_2^2)$, $m_1 (v_1 + v'_1)(v_1 - v'_1) = m_2 (v'_2 + v_2)(v'_2 - v_2)$

$m_1 (v_1 - v'_1) = m_2 (v'_2 - v_2)$, $v_1 + v'_1 = v_2 + v'_2$

$v'_1 - v'_2 = v_2 - v_1$, $m_1 v'_1 + m_2 v'_2 = m_1 v_1 + m_2 v_2$



$$v'_2 = \frac{m_2 - m_1}{m_1 + m_2} v_2 + \frac{2 m_1}{m_1 + m_2} v_1$$

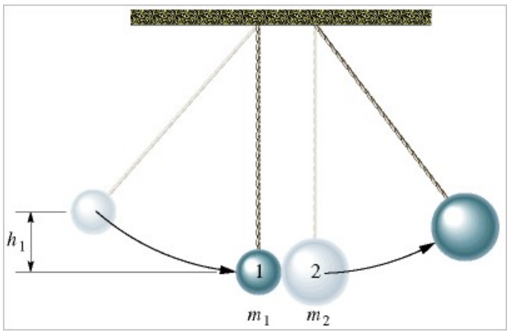
$$v'_1 = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2 m_2}{m_1 + m_2} v_2$$

Special cases: when $v_2 = 0$, $v'_1 = \frac{m_1 - m_2}{m_1 + m_2} v_1$, $v'_2 = \frac{2 m_1}{m_1 + m_2} v_1$

1. Equal masses: $m_1 = m_2$, $v'_1 = 0$, $v'_2 = v_1$
2. A massive target: $m_1 << m_2$, $v'_1 = -v_1$, $v'_2 = \frac{2m_1}{m_2} v_1$
3. A massive projectile: $m_1 >> m_2$, $v'_1 = v_1$, $v'_2 = 2v_1$

Sample Example:

Two metal spheres, suspended by vertical cords, initially just touch. Sphere 1, with mass m_1 = 30 g, is pulled to the left to height h_1 = 8.0 cm, and then released from rest. After swinging down, it undergoes an elastic collision with sphere 2, whose mass m_2 = 75 g. What is the velocity v_{1f} of sphere 1 just after the collision?



$$\frac{1}{2} m_1 v_1^2 = m_1 g h_1, \quad v_1 = \sqrt{2gh_1} = \sqrt{2 \cdot 9.8 \cdot 0.08} = 1.252$$

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2, \quad m_1 v_1^2 = m_1 v'^2_1 + m_2 v'^2_2, \quad m_1 v_1 - m_1 v'_1 = m_2 v'_2,$$

$$m_1 (v_1 + v'_1)(v_1 - v'_1) = m_2 v'^2_2, \quad v_1 + v'_1 = v'_2, \quad 2m_1 v_1 = (m_1 + m_2) v'_2$$

$$v'_2 = \frac{2m_1 v_1}{m_1 + m_2}, \quad v'_1 = \frac{m_1 - m_2}{m_1 + m_2} v_1$$

Sample Example:

A block of mass m_1=1.60 kg initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track collides with a spring attached to a second block of mass m_2=2.10 kg initially moving to the left with a speed of 2.50 m/s. The spring constant is 600 N/m.

(a) Find the velocities of the two blocks after the collision.

$$1.60 \cdot 4.00 + 2.10 \cdot (-2.50) = 1.60 \cdot v_1' + 2.10 \cdot v_2'$$

$$4.00 - (-2.50) = v_2' - v_1', \quad v_2' = 3.12, \quad v_1' = -3.38$$

(b) During the collision, at the instant block 1 is moving to the right with a velocity of +3.00 m/s, determine the velocity of block 2.

$$1.60 \cdot 4.00 + 2.10 \cdot (-2.50) = 1.60 \cdot 3.00 + 2.10 \cdot v_2', \quad v_2' = -1.74$$

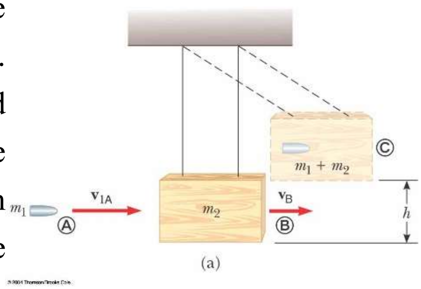
(c) Determine the distance the spring is compressed at that instant.

$$\frac{1}{2} kx^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2} m_1 v_1'^2 - \frac{1}{2} m_2 v_2'^2$$

(d) What is the maximum compression of the spring during the collision? – when two blocks are moving at the same speed.

Example: The Ballistic Pendulum

The ballistic pendulum is an apparatus used to measure the speed of a fast-moving projectile, such as a bullet. A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The bullet imbeds in the block, and the entire system swings through a height h. How can we determine the speed of the bullet from a measurement of h?



Obtain the speed of m_1 and m_2 immediately after collision:

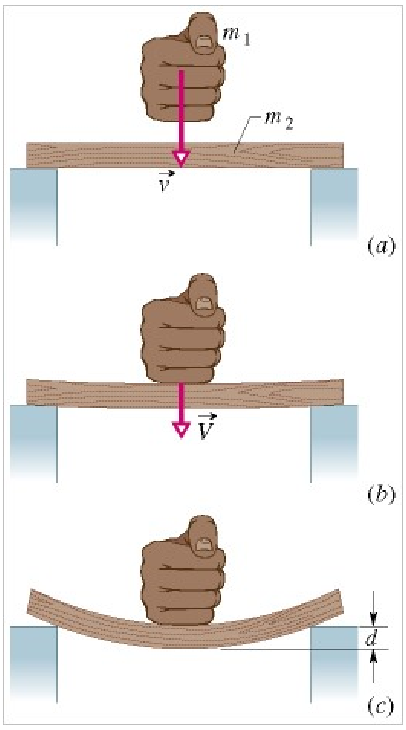
$$v = \sqrt{2gh}$$

Apply the conservation of momentum:

$$(m_1 + m_2)\sqrt{2gh} = m_1 v_{1A}$$

Sample Problem:

A karate expert strikes downward with his fist (of mass m_1 = 0.70 kg), breaking a 0.14 kg board. He then does the same to a 3.2 kg concrete block. The spring constants k for bending are 4.1×10^4 N/m for the board and 2.6×10^6 N/m for the block. Breaking occurs at a deflection d of 16 mm for the board and 1.1 mm for the block.



(a) Just before the object (board or block) breaks, what is the energy stored in it?

Board: $U = \frac{1}{2}kd^2 = \frac{1}{2}4.1 \cdot 10^4 \cdot (16 \cdot 10^{-3})^2 = 5.2J$

Block: $U = \frac{1}{2}kd^2 = \frac{1}{2}2.6 \cdot 10^6 \cdot (1.1 \cdot 10^{-3})^2 = 1.6J$

(b) What is the lowest fist speed v_{fist} required to break the object (board or block)? Assume the following: The collisions are completely inelastic collisions of only the fist and the object. Bending begins just after the collision. Mechanical energy is conserved from the beginning of the bending until just before the object breaks. The speed of the fist and object is negligible at that point.

$$m_1 v_{fist} = (m_1 + m_2)v\,,\quad \frac{1}{2}(m_1 + m_2)v^2 = U \;\rightarrow\; \frac{1}{2}(m_1 + m_2)\left(\frac{m_1 v_1}{m_1 + m_2}\right)^2 = U$$

$$v_{fist} = \frac{1}{m_1}\sqrt{2U(m_1 + m_2)}$$

9.5 Collisions in Two Dimensions

For conservation of momentum:

$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$ → Checked by their components

$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx}\,,\quad p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy}$$

For conservation of kinetic energy:

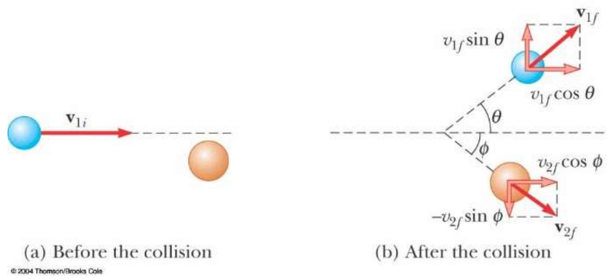
$$K_{1i} + K_{2i} = K_{1f} + K_{2f} \;\rightarrow\; \frac{p_{1i}^2}{2m_1} + \frac{p_{2i}^2}{2m_2} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2} \Rightarrow \frac{p_{1ix}^2 + p_{1iy}^2}{2m_1} + \frac{p_{2ix}^2 + p_{2iy}^2}{2m_2}$$

Simplified questions:

$$m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi$$

$$m_1 v_{1f}\sin\theta = m_2 v_{2f}\sin\phi$$

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$



Example: A proton collides elastically with another proton that is initially at rest. The incoming proton has an intial speed of 3.5x10^5 m/s and makes a glancing collision with the second proton. After collision, one proton moves off at an angle of 37^o to the original direction of motion, and the second deflects at an angle of ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

$$mv_1 = mv'_1\cos 37^o + mv'_2\cos\phi$$
$$mv'_1\sin 37^o = mv'_2\sin\phi$$

$$mv_1^2 = mv'^2_1 + mv'^2_2$$

$$v'_2\cos\phi = v_1 - v'_1\cos 37^o$$

$v'_2 \sin\phi = v'_1 \sin 37^o$

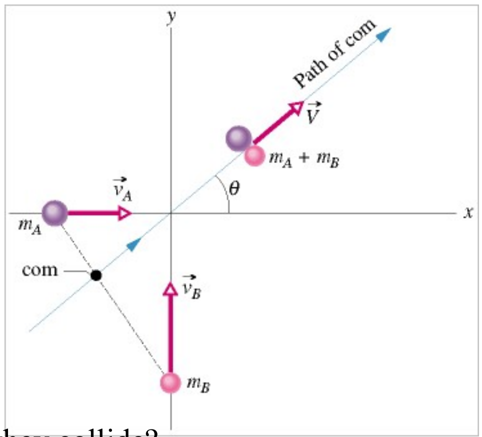
$v'_2{}^2 = v_1{}^2 - v'_1{}^2 = (v_1 - v'_1 \cos 37^o)^2 + (v'_1 \sin 37^o)^2$

$v'_1 = v_1 \cos 37^o = 2.8\times 10^5$

$v'_2 = \sqrt{v_1{}^2 - v'_1{}^2} = \sqrt{35^2 - 28^2}\times 10^4 = 2.1\times 10^5$, $\sin\phi = \dfrac{v'_1}{v'_2}\sin 37^o = \dfrac{4}{3}\dfrac{3}{5} = \dfrac{4}{5}$

Sample Example:

Two skaters collide and embrace, in a completely inelastic collision. Thus, they stick together after impact, where the origin is placed at the point of collision. Alfred, whose mass m_A is 83 kg, is originally moving east with speed v_A = 6.2 km/h. Barbara, whose mass m_B is 55 kg, is originally moving north with speed v_B = 7.8 km/h.



a) What is the velocity $\vec{V}$ of the couple after they collide?

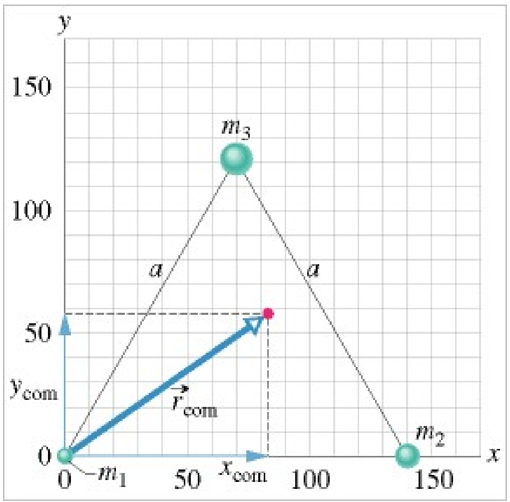
$$\vec{P}_A = \hat{i} m_A v_A = 83\cdot 6.2\hat{i} = 514.6\hat{i}$$

$$\vec{P}_B = \hat{j} m_B v_B = 55\cdot 7.8\hat{j} = 429\hat{j}$$

$$\vec{P} = (m_A + m_B)\vec{v} = \vec{P}_A + \vec{P}_B,\quad \vec{v} = \frac{514.6\hat{i} + 429\hat{j}}{55+83} = 3.72\hat{i} + 3.11\hat{j}$$

$$v = \sqrt{3.72^2 + 3.11^2} = 4.85 km/hr$$

9.6 The Center of Mass



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

Three Dimension:

discrete mass points:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

continuous mass:

$$x_{CM} = \frac{1}{M}\int x dm\,,\quad y_{CM} = \frac{1}{M}\int y dm\,,\quad z_{CM} = \frac{1}{M}\int z dm$$

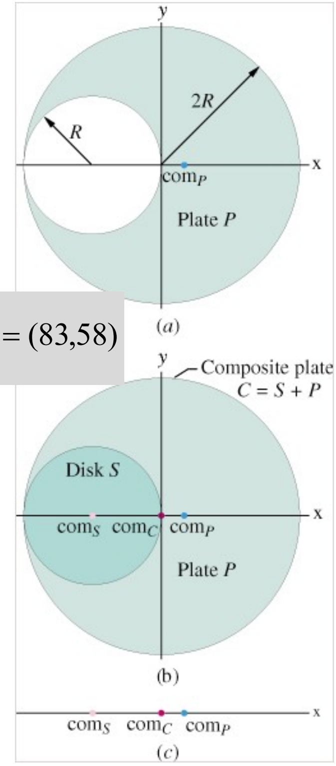
$$x_{CM} = \frac{1}{V}\int x dV\,,\quad y_{CM} = \frac{1}{V}\int y dV\,,\quad z_{CM} = \frac{1}{V}\int z dV$$

Sample Example:

Three particles of masses m_1 = 1.2 kg, m_2 = 2.5 kg, and m_3 = 3.4 kg form an equilateral triangle of edge length a = 140 cm. Where is the center of mass of this three-particle system?

$\vec{r}_1 = (0,0)\,,\ \vec{r}_1 = (140,0)\,,\ \vec{r}_1 = (70,120)$

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{(m_1 + m_2 + m_3)} = \frac{1.2\cdot(0,0) + 2.5\cdot(140,0) + 3.4\cdot(70,120)}{1.2 + 2.5 + 3.4} = (83,58)$$



Sample Example:

Figure shows a uniform metal plate P of radius $2R$ from which a disk of radius R has been stamped out (removed) in an assembly line. Using the xy coordinate system shown, locate the center of mass com$_P$ of the plate.

$$x_{cm} = \frac{4M\cdot 0 + (-M)\cdot(-R)}{4M + (-M)} = \frac{MR}{3M} = \frac{1}{3}R$$

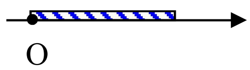
$$x_{cm} = \frac{4M\cdot 0 + (M)\cdot(-R)}{4M + (M)} = -\frac{MR}{5M} = -\frac{1}{5}R$$

Gravitational Potential Energy of a System

Finding the Center of Mass by Integration

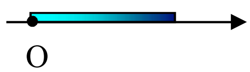
Rod:

Example: Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

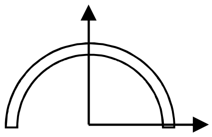


$$X_{CM} = \frac{\sum m_i x_i}{M} = \frac{\int_0^L x(\lambda dx)}{\lambda L} = \frac{1}{2}L$$

Suppose the rod is non-uniform with its mass density varies as $\lambda = \alpha x$. Find the center of mass.



$$M = \int_0^L \lambda dx = \frac{1}{2}\alpha L^2, \quad X_{CM} = \frac{\int_0^L x\lambda dx}{M} = \frac{\alpha \frac{L^3}{3}}{\frac{1}{2}\alpha L^2} = \frac{2}{3}L$$



Semicircular Hoop:

$$dm = \lambda ds = \lambda r d\theta$$
$$\vec{r} = r\cos\theta\hat{x} + r\sin\theta\hat{y}$$

$$M\vec{r}_{CM} = \pi r\lambda\vec{r}_{CM} = \int_0^\pi (r\cos\theta\hat{x} + r\sin\theta\hat{y})\lambda r d\theta$$

$$\vec{r}_{CM} = \frac{2r}{\pi}\hat{y}$$

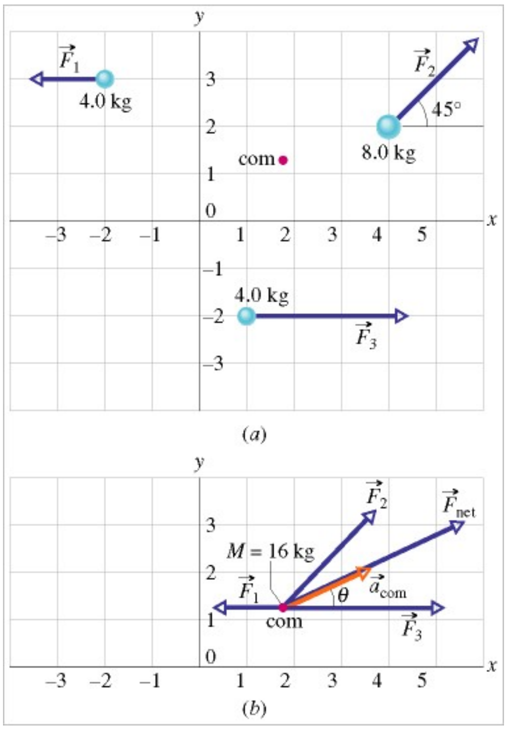
9.7 Motion of a System of Particles

$$M\vec{r}_{CM} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots + m_n\vec{r}_n$$
$$M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n$$

$$M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_n\vec{a}_n$$
$$M\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$$

Sample Example:

The three particles are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are F_1 = 6.0 N, F_2 = 12 N, and F_3 = 14 N. What is the acceleration of the center of mass of the system, and in what direction does it move?



$$\vec{r}_1 = (-2,3), \quad \vec{r}_2 = (4,2), \quad \vec{r}_3 = (1,-2)$$

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

$$\vec{F}_1 = (-6,0), \quad \vec{F}_2 = (6\sqrt{2}, 6\sqrt{2}),$$

$$\vec{F}_3 = (14,0), \quad \vec{F} = (6\sqrt{2}+8, 6\sqrt{2})$$

$$\vec{a} = \frac{\vec{F}}{m_1 + m_2 + m_3} = \frac{(6\sqrt{2}+8)\cdot\hat{i} + 6\sqrt{2}\cdot\hat{j}}{16}$$

Kinetic Energy of a System

$$K = \sum K_i = \sum \frac{1}{2} m_i v_i^2 , \qquad \vec{v}_i = \vec{v}_{CM} + \vec{u}_i$$

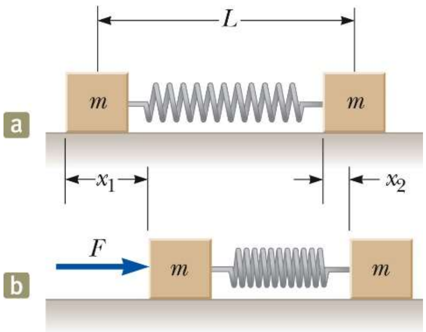
$$K = \frac{1}{2}\sum m_i(\vec{v}_{CM} + \vec{u}_i)\cdot(\vec{v}_{CM} + \vec{u}_i) = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\sum m_i u_i^2 = \frac{1}{2}Mv_{CM}^2 + K_{rel}$$

The Center-of-Mass Reference Frame

If the net external force on a system remains zero, the velocity of the center of mass remains constant.

$$\vec{F}_{total} = 0 \;\rightarrow\; \frac{d}{dt}(m\vec{v}_{CM}) = \vec{F}_{total} = 0 \;\rightarrow\; \vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots}{m_1 + m_2 + \ldots}$$

9.8 Deformable Systems



Pushing on a Spring:

As shown in the right figure, two blocks are at rest on a frictionless, level table. Both blocks have the same mass m, and they are connected by a spring of negligible mass. The separation distance of the blocks when the spring is relaxed is L. During a time interval Δt, a constant force of magnitude F is applied horizontally to the left block, moving it through a distance x_1. During the time interval, the right block moves through a distance x_2. At the end of this time interval, the force F is removed. (a) Find the resulting speed v_{cm} of the center of mass of the system.

The external force is used to increase the momentum of the system. The momentum is related to the final speed of the system.

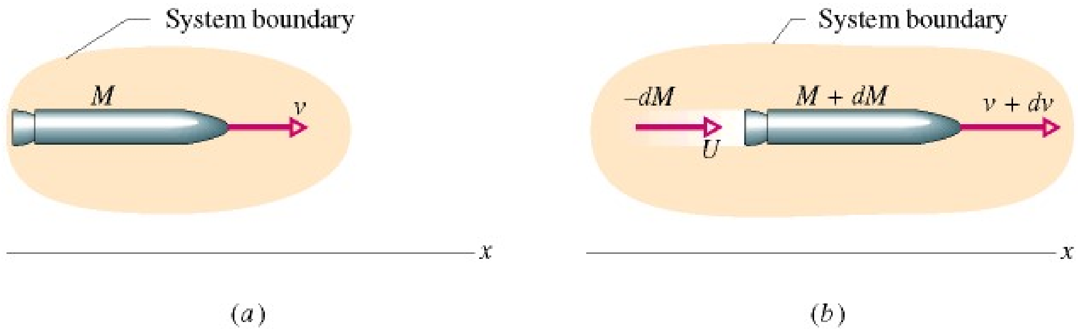
1. $F\Delta t = (m+m)v_{CM}$ (change of the momentum)

The displacement of the center of mass (final com position is subtracted by initial com position) is calculated by the average speed of the com.

2. $v_{CM,avg}\Delta t = \frac{x_1 m + (L + x_2)m}{m+m} - \frac{0 + mL}{m+m} = \frac{1}{2}(x_1 + x_2) \;\rightarrow\; v_{CM,avg} = \frac{1}{2}(0 + v_{CM})$

$$F\frac{x_1+x_2}{v_{CM}}=(m+m)v_{CM} \rightarrow v_{CM}=\sqrt{\frac{F(x_1+x_2)}{2m}}$$

9.9 Rocket Propulsion



$P_j = P_i = Mv$, $Mv = -dM\cdot U + (M+dM)(v+dv)$

v_{rel} : relative speed between the rocket and the exhaust products

$v_{rel} = v + dv - U$

$-dM\cdot U + Mdv + (v+dv)dM = 0 \rightarrow dM\cdot(v+dv-U) + Mdv = 0$

$v_{rel}dM + Mdv = 0$, $-v_{rel}dM = Mdv$

(a) $-v_{rel}\frac{dM}{dt} = M\frac{dv}{dt}$, $\frac{dM}{dt} = -R$, $Rv_{rel} = Ma$

(b) $dv = -v_{rel}\frac{dM}{M}$, $\int_{vi}^{vf} dv = -v_{rel}\int_{Mi}^{Mf}\frac{dM}{M}$, $v_f = v_i - v_{rel}\ln\frac{M_f}{M_i} = v_i + v_{rel}\ln\frac{M_i}{M_f}$

Sample Problem:

A rocket whose initial mass M_i is 850 kg consumes fuel at the rate R = 2.3 kg/s. The speed v_{rel} of the exhaust gases relative to the rocket engine is 2800 m/s.

a) What thrust does the rocket engine provide?

$T = Rv_{rel} = 2.3\cdot 2800 = 6440$

b) What is the initial acceleration of the rocket?

$a = \frac{T}{M} = \frac{6440}{850} = 7.6m/s$

(c) Suppose, instead, that the rocket is launched from a spacecraft already in deep space, where we can neglect any gravitational force acting on it. The mass M_f of the rocket when its fuel is exhausted is 180 kg. What is its speed relative to the spacecraft at that time? Assume that the spacecraft is so massive that the launch does not alter its speed.

$$v_0 = 0\,,\quad v_f = v_{rel} \ln \frac{M_i}{M_f} = 2800 \cdot \ln \frac{850}{180} = 4300 m/s$$