Lecture 15 Oscillation Motion

Periodic motions:

A special kind of periodic motion occurs when the force that acts on a particle is proportional to the displacement of the particle from the equilibrium position and is always directed toward the equilibrium position. When this type of force acts on a particle, the particle exhibits simple harmonic motion which will serve as an analysis model for a large class of oscillation problems.

15.1 Motion of an Object Attached to a Spring

HOOK'S LAW: restoring force

$$F_r = -kx$$

NEWTON'S 2^{ND} LAW:

$$F = ma \rightarrow ma = -kx$$

When a particle is under the effect of a linear restoring force, the motion it follows is a special type of oscillatory motion called simple harmonic motion.

$$-kx = ma \rightarrow a_x = -\frac{k}{m}x$$

That is, the acceleration is proportional to the displacement of the particle from equilibrium and is in the opposite direction.

15.2 The Particle in Simple Harmonic Motion

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

let $\omega^2 = \frac{k}{m}$, $\frac{d^2x}{dt^2} = -\omega^2 x$

→ $\frac{d^2x}{dt^2} + \omega^2 x = 0$

$x(t) = A\cos(\omega t + \phi)$

$\frac{dx(t)}{dt} = -A \cdot \omega \cdot \sin(\omega t + \phi)$

$\frac{d^2x(t)}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$

→ $\omega = \sqrt{\frac{k}{m}}$, ϕ is called phase constant

$T = \frac{2\pi}{\omega}$, $T = 2\pi\sqrt{\frac{m}{k}}$

$v(t) = \frac{dx(t)}{dt} = -\omega A\sin(\omega t + \phi)$ → $|v_{\max}| = \omega A = \sqrt{\frac{k}{m}}A$

$a(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$ → $|a_{\max}| = \omega^2 A = \frac{k}{m}A$

initial condition to obtain A and ϕ:

1. $x(0) = A\cos\phi = A$, $v(0) = -A\omega\sin(\phi) = 0$ → A=? ϕ=?

2. $x(0) = A\cos\phi = 0$, $v(0) = -\omega A\sin\phi = v_i$ → A=? ϕ=?

Example: A block-spring system

A block with a mass of 200 g is connected to a light horizontal spring of force constant 5 N/m and is free to oscillate on a horizontal, frictionless surface. (a) if the block is displaced 5 cm from equilibrium and released from rest, find the period of its motion.

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{0.2}} = 5$, $T = \frac{2\pi}{\omega} = 1.26s$

(b) Determine the maximum speed and the maximum acceleration of the block

$x(t) = A\cos(\omega t + \phi)$, $x(0) = A\cos\phi = 5cm$, $v(0) = -\omega A\sin\phi = 0$

-> $\phi = 0$, $A = 5cm = 0.05m$

$v_{\max} = \omega A = 0.25 m/s$, $a_{\max} = \omega^2 A = 1.25 m/s^2$

Example: Initial condition

Suppose that the initial position x_i and the initial velocity v_i of a harmonic oscillator of known angular frequency are given: that is $x(0)=x_i$, $v(0)=v_i$. Find general expression for the amplitude and the phase constant in terms of these initial parameters.

$$x_i = A\cos\phi\,,\quad v_i = -\omega A\sin\phi\,,\quad \tan\phi = -\frac{v_i}{\omega x_i}\,,\quad A = \sqrt{{x_i}^2 + (\frac{v_i}{\omega})^2}$$

Astronaut Alan L. Bean measures his body mass during the second Skylab mission by sitting in a seat attached to a spring and oscillating back and forth.

15.3 Energy of the Simple Harmonic Oscillator

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}kA^2 \sin^2(\omega t + \phi)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

$$E = K + U = \frac{1}{2}kA^2$$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2\,,\quad v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{A^2 - x^2}$$

Example: Oscillations on a horizontal surface

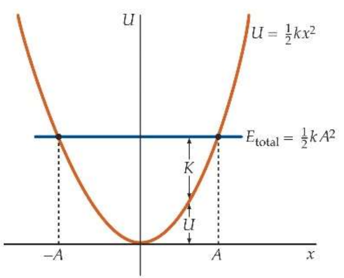
A 0.5 kg object connected to a massless spring of force constant 20 N/m oscillates on a horizontal, frictionless track. (a) Calculate the total energy of the system and the maximum velocity of the object if the amplitude of the motion is 3 cm.

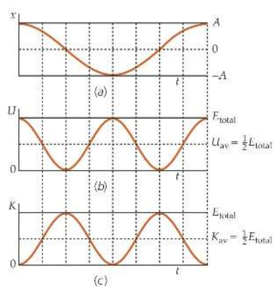
$$A = 0.03m\,,\quad E = \frac{1}{2}kA^2 = \frac{1}{2}20\cdot(0.03)^2 = 0.009J$$

$$v_{\max} = \omega A = \sqrt{\frac{20}{0.5}}0.03 = 0.19m/s$$

(b) What is the velocity of the object when the position is equal to 2 cm?

$$v = \omega\sqrt{A^2 - x^2} = \sqrt{\frac{20}{0.5}}\sqrt{0.03^2 - 0.02^2} = 0.141m/s$$

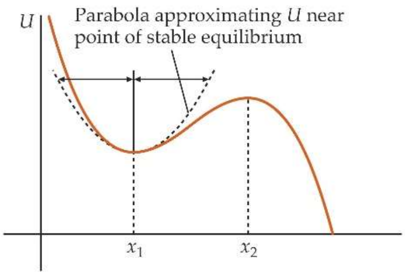




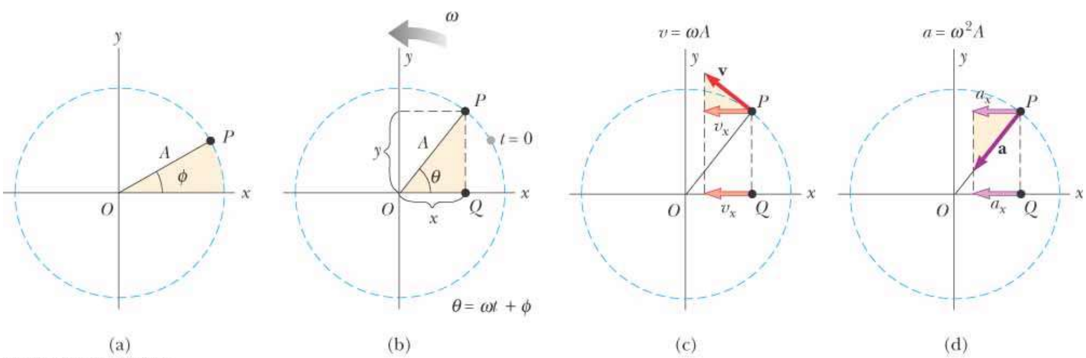
General Motion Near Equilibrium

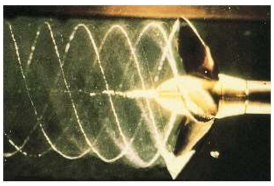
$$U = A + B(x - x_1)^2$$

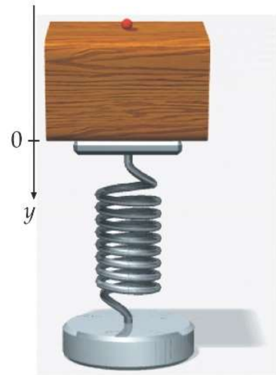
$$F = -\frac{dU}{dx} = -2B(x - x_1) = -k(x - x_1)$$



15.4 Comparing Simple Harmonic Motion with Uniform Circular Motion





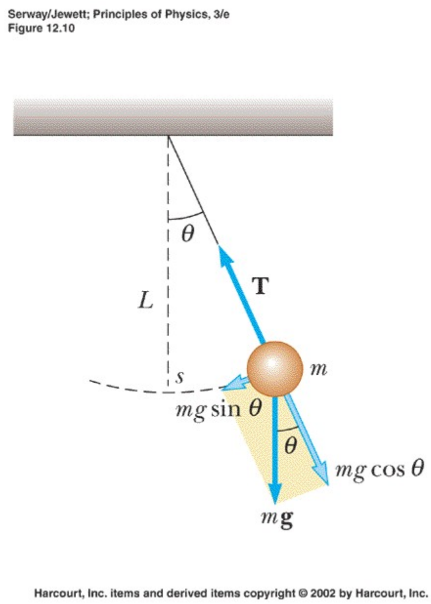


Some Oscillating Systems

Example: A block attached to a spring oscillates vertically with a frequency of 4 Hz and an amplitude of 7 cm. A tiny bead is placed on top of the oscillating block just as it reaches its lowest point. Assume that the bead's mass is so small that its effect on the motion of the block is negligible. At what distance from the block's equilibrium position does the bead lose contact with the block?

$$mg = -ky \rightarrow g = -\omega^2 y \rightarrow y = -\frac{g}{\omega^2} = -\frac{9.8}{(2\pi \times 4)^2} = -0.0155 \ \text{m}$$

15.5 The Pendulum



Example: Pendulum

$$m\frac{d^2 s}{dt^2} = -mg\sin\theta$$

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

for a small θ, $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$

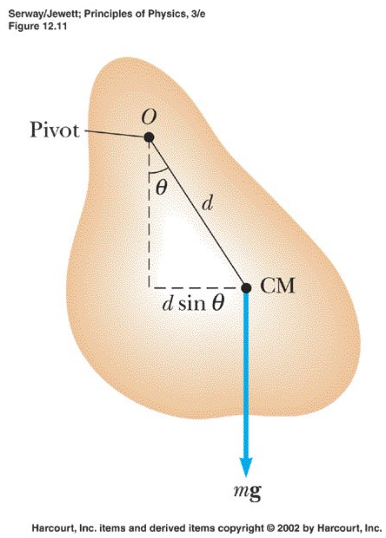
$$\omega = \sqrt{\frac{g}{L}}, \quad T = 2\pi\sqrt{\frac{L}{g}}$$

Example: A measure of height

A man enters a tall tower, needing to know its height. He notes that a long pendulum extends from the ceiling almost to the floor and that its period is 12 s. How tall is the tower?

$$L = \frac{T^2}{4\pi^2}g = \frac{12^2}{4\pi^2}9.8 = 35.7m$$

Physical Pendulum



$$\tau = -mgd\sin\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

where I is relative to O axis not I_{cm}

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\sin\theta$$

for a small θ, $\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta$

$$\omega = \sqrt{\frac{mgd}{I}}, \quad T = 2\pi\sqrt{\frac{I}{mgd}}$$

Example: A swinging sign

A circular sign of mass M and radius R is hung on a nail from a small loop located at one edge. After it is placed on the nail, the sign oscillates in a vertical plane. Find the period of oscillation if the amplitude is small.

$$I = I_{cm} + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

$$T = 2\pi\sqrt{\frac{\frac{3}{2}MR^2}{MgR}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Example: Torsional Pendulum

A rigid object suspended by a wire attached at the top to a fixed support. When the object is twisted through some angle θ, the twisted wire exerts on the object a restoring torque that is proportional to the angular position. That is, $\tau = -\kappa\theta$, where κ is called the torsion constant of the support wire. Find the period of oscillation.

$$\tau = I\alpha = -\kappa\theta \rightarrow I\frac{d^2\theta}{dt^2} + \kappa\theta = 0$$

$$\omega = \sqrt{\frac{\kappa}{I}} \rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

15.6 Damped Oscillations

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}, \boxed{\text{assume } x = Ae^{at}}, \quad mAa^2e^{at} + bAae^{at} + kAe^{at} = 0$$

$$ma^2 + ba + k = 0, \quad a = \frac{-b}{2m} \pm \frac{\sqrt{b^2 - 4mk}}{2m} = \frac{-b}{2m} \pm i\sqrt{\frac{k}{m} - (\frac{b}{2m})^2}$$

$$x = Ae^{-\frac{b}{2m}t}e^{i\sqrt{\frac{k}{m}-(\frac{b}{2m})^2}t} + Be^{-\frac{b}{2m}t}e^{i\sqrt{\frac{k}{m}-(\frac{b}{2m})^2}t} = e^{-\frac{b}{2m}t}\left[Ae^{i\sqrt{\frac{k}{m}-(\frac{b}{2m})^2}t} + Be^{-i\sqrt{\frac{k}{m}-(\frac{b}{2m})^2}t}\right]$$

$$x = e^{-\frac{b}{2m}t}\left[A\cos(\omega t) + iA\sin(\omega t) + B\cos(\omega t) - iB\sin(\omega t)\right], \quad \omega^2 = \frac{k}{m} - (\frac{b}{2m})^2$$

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$$x = e^{-\frac{b}{2m}t}\left[(A+B)\cos(\omega t) + i(A-B)\sin(\omega t)\right] = e^{-\frac{b}{2m}t}\left[C\cos(\omega t) + D\sin(\omega t)\right]$$

$$= \sqrt{C^2 + D^2}\, e^{-\frac{b}{2m}t}\left[\frac{C}{\sqrt{C^2+D^2}}\cos(\omega t) + \frac{D}{\sqrt{C^2+D^2}}\sin(\omega t)\right]$$

$$= A_0 e^{-bt/2m}\left[\sin(\phi)\cos(\omega t) + \cos(\phi)\sin(\omega t)\right] = A_0 e^{-bt/2m}\sin(\omega t + \phi)$$

15.7 Forced Oscillations

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 \sin\omega t\ , \ \boxed{\text{assume} \quad x = A\sin(\omega t + \phi)}$$

$$-m\omega^2 A\sin(\omega t + \phi) + kA\sin(\omega t + \phi) + b\omega A\cos(\omega t + \phi) = F_0 \sin\omega t$$

$$(k - m\omega^2)A\sin(\omega t + \phi) + b\omega A\cos(\omega t + \phi) = F_0 \sin\omega t$$

$$(\frac{k}{m} - \omega^2)A(\sin\phi\cos\omega t + \cos\phi\sin\omega t) + \frac{b}{m}\omega A(\cos\phi\cos\omega t - \sin\phi\sin\omega t) = \frac{F_0}{m}\sin\omega t$$

$$(\frac{k}{m} - \omega^2)\sin\phi = -\frac{b}{m}\omega\cos\phi \ \rightarrow \ (\omega^2 - \frac{k}{m})\sin\phi = \frac{b}{m}\omega\cos\phi$$

$$\sin\phi = \frac{b\omega/m}{\sqrt{(\omega^2 - \frac{k}{m})^2 + (\frac{b}{m}\omega)^2}}\ , \quad \cos\phi = \frac{\left(\omega^2 - k/m\right)}{\sqrt{(\omega^2 - \frac{k}{m})^2 + (\frac{b}{m}\omega)^2}}$$

$$(\frac{k}{m} - \omega^2)A\cos\phi - \frac{b}{m}\omega A\sin\phi = \frac{F_0}{m} \ \rightarrow \ (\omega^2 - \frac{k}{m})A\cos\phi + \frac{b}{m}\omega A\sin\phi = -\frac{F_0}{m}$$

$$A = -\frac{F_0/m}{(\omega^2 - \frac{k}{m})\cos\phi + \frac{b}{m}\omega\sin\phi}$$

$$A = -\frac{F_0/m}{\sqrt{(\omega^2 - \frac{k}{m})^2 + \left(\frac{b\omega}{m}\right)^2}}$$

$$x = -\frac{F_0/m}{\sqrt{(\omega^2 - \frac{k}{m})^2 + \left(\frac{b\omega}{m}\right)^2}}\sin(\omega t + \phi)\ , \quad \sin\phi = \frac{b\omega/m}{\sqrt{(\omega^2 - \frac{k}{m})^2 + (\frac{b}{m}\omega)^2}}$$