Chapter 32 Inductance

32.1 Self-Induction and Inductance

Self-Inductance

$\Phi_m = BA = \mu_0 nIA \propto I$ --> $\Phi_m = LI$

The unit of the inductance is henry (H). $1\text{H} = 1\frac{\text{Wb}}{\text{A}} = 1\frac{\text{T}\cdot\text{m}^2}{\text{A}}$

When the current in the circuit is changing, the magnetic flux is also changing.

$\frac{d\Phi}{dt} = \frac{d(LI)}{dt} = L\frac{dI}{dt}$ --> The induced emf should be $\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$

The self inductance of a infinite long solenoid:

$\Phi_m = NAB = NA\mu_0\frac{N}{l}I = \mu_0 A\frac{N^2}{l}I$ --> $L = \frac{\Phi_m}{I} = \mu_0 A\frac{N^2}{l}$

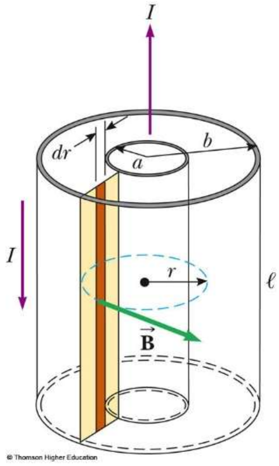
$$\varepsilon = -L\frac{dI}{dt}$$

Considering the inductor having an internal resistance r, the potential difference is:

$$\Delta V = \varepsilon - Ir = -L\frac{dI}{dt} - Ir$$

Example: Model a long coaxial cable as two thin, concentric, cylindrical conducting shells of radii *a* and *b* and length *l*. The conducting shells carry the same current in opposite directions. Calculate the inductance *L* of this cable.

Calculate magnetic flux:

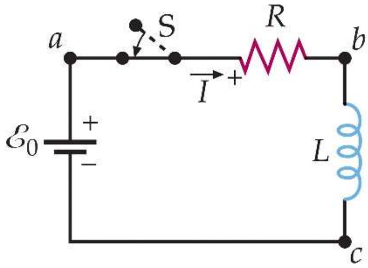


$$B = \frac{\mu_0 I}{2\pi r} \rightarrow \Phi = \int_0^l \int_a^b \frac{\mu_0 I}{2\pi r} dr dz = \frac{\mu_0 I}{2\pi} l \ln\left(\frac{b}{a}\right)$$

$$\Phi = LI \rightarrow L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$$

32.2 RL Circuits

Use Kirchhoff's rule:



$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$\varepsilon_0 - IR - L\frac{dI}{dt} = 0$$

Differential Eq: $\varepsilon_0 - IR - L\frac{dI}{dt} = 0$ --> $\varepsilon_0 - IR = L\frac{dI}{dt}$

--> $dt = L\frac{dI}{(\varepsilon_0 - IR)}$ --> $dt = -\frac{L}{R}\frac{d(\varepsilon_0 - IR)}{(\varepsilon_0 - IR)}$

--> $-\frac{R}{L}t = \ln\left(\frac{\varepsilon_0 - IR}{\varepsilon_0}\right)$ --> $\varepsilon_0 - IR = \varepsilon_0 e^{-\frac{R}{L}t}$ --> $I = \frac{\varepsilon_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

Time constant: $\tau = \frac{L}{R}$

Example: Find the total energy dissipated in the resistor R, when the current in the inductor decreases from its initial value of I_0 to 0?

$$I = \frac{\varepsilon_0}{R}e^{-\frac{R}{L}t},\quad P = I^2 R \;\text{-->}\; U = \int_0^\infty I_0^2 e^{-2\frac{R}{L}t} R\,dt = \frac{1}{2}LI_0^2$$

32.3 Energy in a Magnetic Field

Obtain the magnetic energy from the emf induced by self inductance.

$\Phi_m = LI$ --> The induced emf is $\varepsilon = -\frac{d\Phi_m}{dt} = -L\frac{dI}{dt}$

The energy dissipated or the power is $P = IV = I\varepsilon = -LI\frac{dI}{dt}$

The total energy when the current has reached its final value I_f is:

$$U = \int_{t=0}^{t=tf}\left(LI\frac{dI}{dt}\right)dt = \int_{I=0}^{I=If} LI\,dI = \frac{1}{2}LI^2$$

Calculate the magnetic energy by obtaining the energy stored in the self inductor of an infinite solenoid.

$B = \mu_0 nI$, $\Phi = nlA(\mu_0 nI) = LI$

$U = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 (Al) I^2 = u_m V$

$u_m = \frac{1}{2}\mu_0 n^2 I^2 = \frac{B^2}{2\mu_0}$ <----> $u_e = \frac{1}{2}\varepsilon_0 E^2$ (Do you remember how to get this?)

Example: A certain region of space contains a uniform magnetic field of 0.020 T and a uniform electric field of 2.5 X 10^6 N/C. Find (a) the total electromagnetic density.

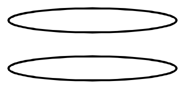
$$u_e = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\left(8.85\times10^{-12}\right)\left(2.5\times10^6\right)^2 = 27.7\text{J/m}^3$$

$$u_m = \frac{B^2}{2\mu_0} = \frac{(0.02)^2}{2\left(4\pi\times10^{-7}\right)} = 159\text{J/m}^3$$

<u>32.4 Mutual Inductance</u>

Mutual Inductance

The magnetic field of loop1 is: $B \sim \frac{\mu_0 I_1}{2R}$



The flux at 2 is $\Phi_2 \sim \frac{\mu_0 I_1}{2R}\pi R^2 = \mu_0 \frac{\pi R^2}{2} I_1 = I_1 M_{12}$

The flux at 1 is $\Phi_1 \sim \mu_0 \frac{\pi R^2}{2} I_2 = I_2 M_{21}$

The concept of inductance: $\Phi = MI$

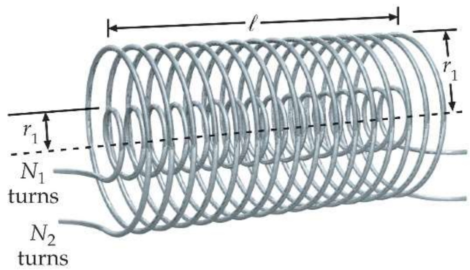
$M_{12} = M_{21}$ --> The mutual inductance is determined when the geometrical configuration between the two loops is given.

B in 1 due to 2: $B = \mu_0 \frac{N_2}{l} I_2$

Flux in 1: $\Phi = N_1\left(\pi r_1^2\right)\left(\mu_0 \frac{N_2}{l} I_2\right) = I_2 M_{21}$

B in 2 due to 1: $B = \mu_0 \frac{N_1}{l} I_1$

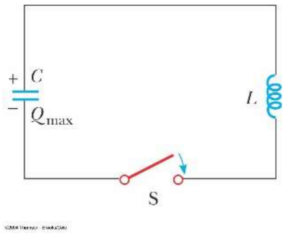
Flux in 2: $\Phi = N_2\left(\pi r_1^2\right)\left(\mu_0 \frac{N_1}{l} I_1\right) = I_1 M_{12}$



$$M_{12} = M_{21} = \mu_0 \left(\pi r_1^2\right) \frac{N_1 N_2}{l}$$

32.5 Oscillations in an LC Circuit

Kirchhoff's Rule:



$$-L\frac{dI}{dt} - \frac{Q}{C} = 0 \rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

Compare with: $F = ma = -kx$

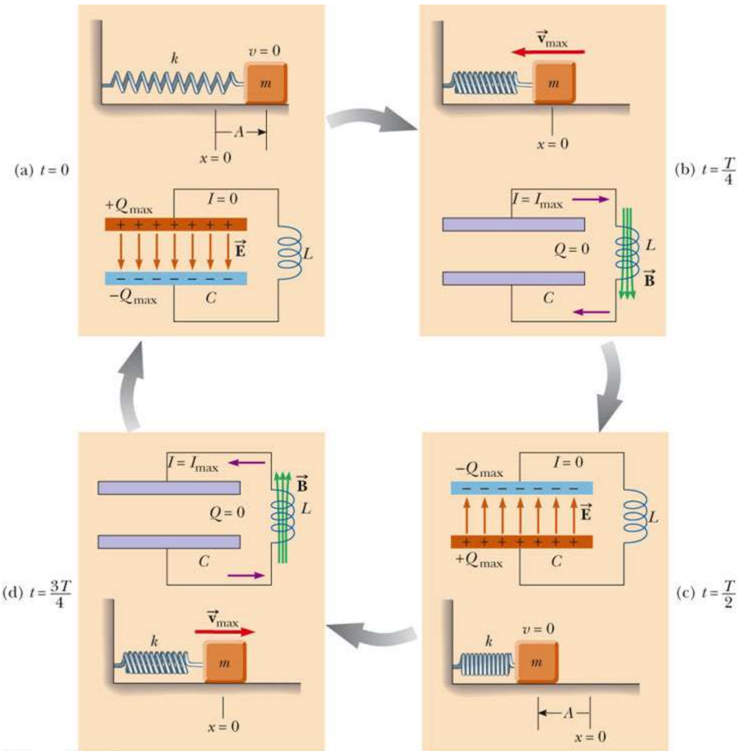
$$m\frac{d^2x}{dt^2} + kx = 0 \quad \text{solution:} \quad x(t) = A\cos(\omega t + \phi)$$

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \quad \text{solution:} \quad Q(t) = Q_{\max}\cos(\omega t + \phi), \quad \omega = \frac{1}{\sqrt{LC}}$$

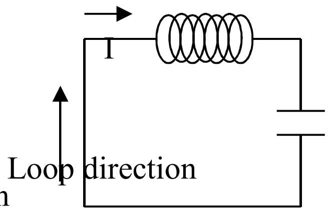
$$\rightarrow \quad I = -\omega Q_{\max}\sin(\omega t + \phi)$$

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{L}{2}I^2 = \frac{1}{2C}Q_{\max}^2\cos^2(\omega t + \phi) + \frac{L}{2}\omega^2 Q_{\max}^2 \sin^2(\omega t + \phi)$$

$$= \frac{Q_{\max}^2}{2C}$$



Apply the Kirchhoff's loop rule:



$$-L\frac{dI}{dt}-\frac{Q}{C}=0$$

--> $L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$ (2nd Differential Equation, compare with harmonic oscillation: $F=ma=m\frac{d^2x}{dt^2}=-kx$ with the answer of $x=A\cos(\omega t+\delta)$, where $\omega=\sqrt{\frac{k}{m}}$)

Remember the pattern of this differential equation:

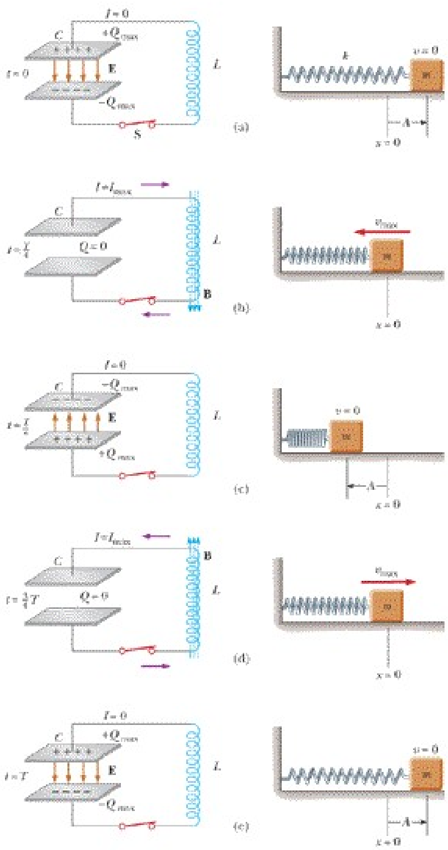
$L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$ --> the solutions are periodical functions and the useable functions are $\sin(x)$ $(\cos(x))$, $\exp(ix)$ $(\exp(x))$, …

Guess that the answer is $Q=A\cos(Bt+C)$. (Here A and C can be determined by initial conditions.)

--> $-LB^2A\cos(Bt+C)+\frac{1}{C}A\cos(Bt+C)=0$ --> $B=\frac{1}{\sqrt{LC}}\equiv\omega$

--> $Q=A\cos(\omega t-\delta)$ & $I=\frac{dQ}{dt}=-\omega A\sin(\omega t-\delta)=-I_{peak}\sin(\omega t)$

Serway/Jewett; Principles of Physics, 3/e
Figure 24.9



Capacitor --> Electric Field --> Potential Energy

Inductor --> Moving of Charges --> Kinetic Energy

The average energy stored in the capacitor (inductor) is $\frac{Q^2}{2C}$ $(\frac{1}{2}LI^2)$.

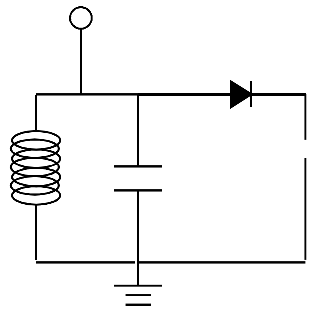
The instantaneous energy transferring in the circuit is:

$$\frac{Q^2}{2C}+\frac{1}{2}LI^2=\frac{A^2\cos^2(\omega t-\delta)}{2C}+\frac{1}{2}L\omega^2A^2\sin^2(\omega t-\delta)=\frac{A^2}{2C}=\frac{Q_{peak}^2}{2C}=\frac{1}{2}LI_{peak}^2$$

What are physical pictures of Q_{peak} and I_{peak}?

Example: A 2-μF capacitor is charged to 20 V and the capacitor is then connected across a 6-μH inductor. (a) What is the frequency of oscillation? (b) What is the peak value of the current?

(a) $\omega=\frac{1}{\sqrt{LC}}$, (b) $\frac{CV^2}{2}=\frac{Q_{peak}^2}{2C}=\frac{LI_{peak}^2}{2}$



Simple AM Radio receiver:

32.6 The RLC Circuit

Kirchhoff's Rule:

$$-IR-L\frac{dI}{dt}-\frac{Q}{C}=0$$

Power Consideration:

$$P=LI\frac{dI}{dt}+\frac{Q}{C}I=-I^2R$$

$$\rightarrow\ L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0$$

Compare with damped oscillation:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0\ \rightarrow\ x(t)=Ae^{nt},\ mn^2+bn+k=0\ \rightarrow\ n=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$$

$$b^2<4mk\ \rightarrow\ x(t)=Ae^{-\frac{b}{2m}t}\cos\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)$$

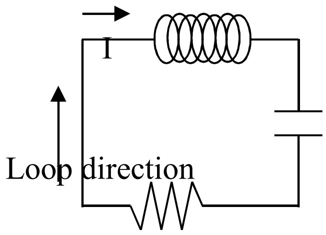
$$Q(t)=Q_{\max}e^{-\frac{R}{2L}t}\cos\left(\frac{\sqrt{4L/C-R^2}}{2L}t\right)$$

TABLE 32.1

Analogies Between Electrical and Mechanical Systems

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped object on a spring

RLC Circuit (Damped Oscillation)



Diff Eq: $-L\frac{dI}{dt} - \frac{Q}{C} - IR = 0$

--> $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$ (Damped Oscillations: $F = ma = -kx - bv$)

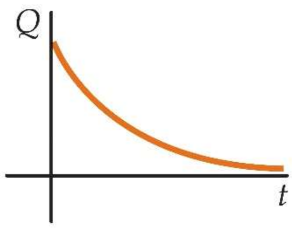
The natural frequency (no resistaor) is: $\omega_0 = \frac{1}{\sqrt{LC}}$ ($L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$).

We guess a solution of $Q = Ae^{Bt}$ for solving the differential equation

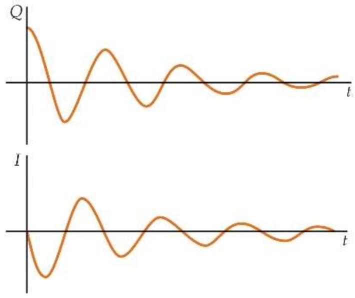
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

$$\left(LB^2 + RB + \frac{1}{C}\right)Ae^{Bt} = 0 \quad \text{-->} \quad B = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

over-damped: $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$



under-damped: $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$



under-damped solution: $Q = Ae^{-\frac{R}{2L}t} e^{\pm i\sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}\,t}$

The energy distributed in the circuit elements is:

$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$ --> $LI\frac{dI}{dt} + RI^2 + \frac{Q}{C}\frac{dQ}{dt} = 0$

--> $\frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{Q^2}{2C}\right) + I^2R = 0$